Undecidability of Quantized State Feedback Control for Discrete Time Linear Hybrid Systems

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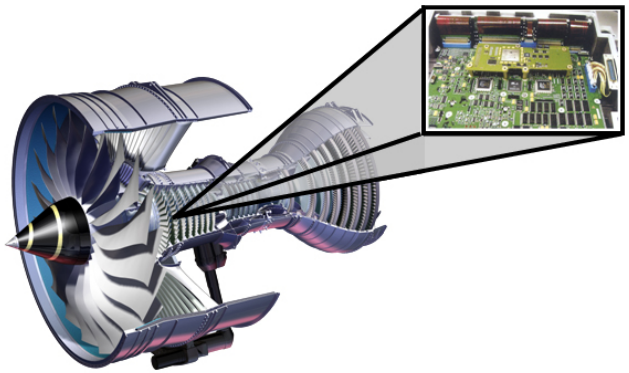
Outline

Embedded Systems

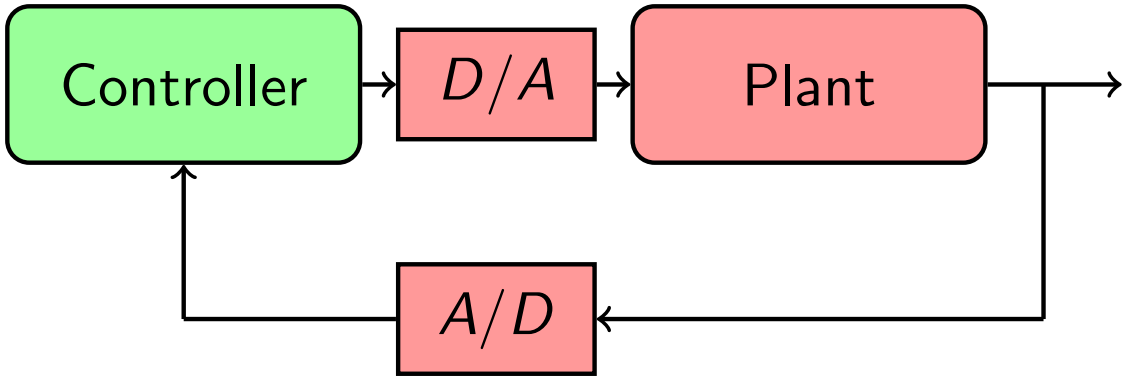
Examples

Wikipedia: An *embedded system* is a computer system designed to do a few dedicated functions with real-time computing constraints



Embedded Systems

modelled as Hybrid Systems



every T seconds **do** // *sampling time*
$\hat{x} = AnalogToDigital(read(\text{plantState}))$
try {
$\hat{u}$=$ctrLaw(\hat{x})$
$send(DigitalToAnalog(\hat{u}))$
} **catch** $(notInCtrReg(\hat{x}))$
{ $FDIR(\hat{x})$ } // *fault isolation and recovery*

Control Software Synthesis (Ideally)

Model Based Design [Henzinger, Sifakis, 2006]

Specifications are easier to define than control software

Input:

- Plant modelled as a (discrete time) **Hybrid System**
- Closed Loop System Level Specifications (**Safety** + **Liveness**)
- Implementation Specifications (**WCET**, **quantization**, etc.)

Output:

- **Correct-by-construction** automatically generated control software
- Guaranteed non functional requirements (WCET)
- **Robustness** (wrt plant parameter variations or disturbances)

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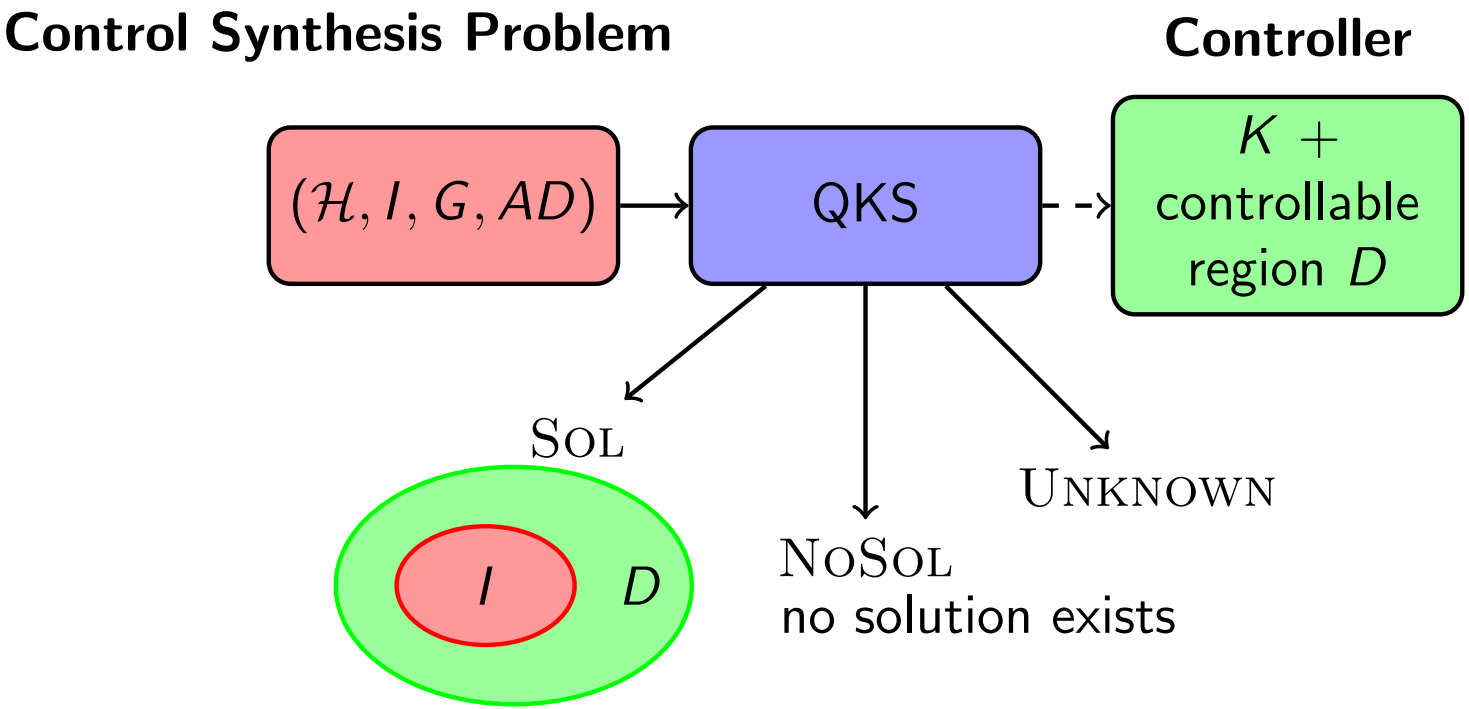
Controller Synthesis for DTLHSs

The tool QKS [Mari, Melatti, Salvo, Tronci, CAV 2010, EMSOFT 2012, CDC 2012]



UNKNOWN stems from undecidability of the problem

Outline

Discrete Time Linear Hybrid Systems (DTLHSs)

A **Discrete Time Linear Hybrid System (DTLHS)** $\mathcal{H}$ is a tuple (X, U, Y, N) where:

- X is a finite sequence of **present state variables**. *Next state* variables X' are obtained by decorating with $'$ all variables in X.
- U is a finite sequence of **input variables**, that models *controllable inputs*.
- Y is a finite sequence of **auxiliary variables** that models *modes* or *uncontrollable inputs* (e.g., *disturbances*).
- $N(X, U, Y, X')$ is a linear predicate over $X \cup U \cup Y \cup X'$ defining the **transition relation** (*next state*) of the system.

Each variable $w \in W$ range over a bounded or unbounded integer or real interval $\mathcal{D}_w$. $D_W = \prod_{w \in W} D_w$.

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DTLHS Semantics

as Labeled Transition Systems (LTS)

A **Labeled Tranisition System (LTS)** $\mathcal{S}$ is a tuple (S, A, T)

- S is a possibly infinite set of **states**,
- A is a possibly infinite set of **actions**
- $T : S \times A \times S \to \mathbb{B}$ is the **transition relation** of $\mathcal{S}$.

A **run** for $\mathcal{S}$ is a sequence $\pi = s_0, a_0, s_1, a_1, s_2, a_2, \ldots$ of states s_t and actions a_t s. t. $\forall t \geq 0 \; T(s_t, a_t, s_{t+1})$.

The dynamics of $\mathcal{H}$ is defined by $\mathrm{LTS}(\mathcal{H}) = (\mathcal{D}_X, \mathcal{D}_U, \bar{N})$ where: $\bar{N} : \mathcal{D}_X \times \mathcal{D}_U \times \mathcal{D}_X \to \mathbb{B}$ is a function s.t. $\bar{N}(x, u, x') = \exists\, y \in \mathcal{D}_Y \; N(x, u, y, x')$.

Controller and Closed Loop System

A formal definition

A **controller** restricts the dynamics of an LTS $\mathcal{S}$ so that all states an *initial region I* will reach in one or more steps a givern *goal region G* (**Liveness Specifications**).

A *controller* for $\mathcal{S}$ is a function $K : S \times A \to \mathbb{B}$ such that $\forall s \in S$, $\forall a \in A$, if $K(s, a)$ then $\exists s'\ T(s, a, s')$.

$\mathcal{S}^{(K)}$ denotes the **closed loop system**, that is the LTS $(S, A, T^{(K)})$, where $T^{(K)}(s, a, s') = T(s, a, s') \wedge K(s, a)$. (i.e. the plant in parallel with the controller)

LTS Reachability and Control Problem

A **reachability problem** and a **control problem** are a triple $(\mathcal{S}, I, G)$, where: $\mathcal{S}$ is an LTS (S, A, T) and $I, G \subseteq S$.

G is **reachable** from I if there exists a run π of $\mathcal{S}$ such that $\pi^{(S)}(0) \in I$ and $\pi^{(S)}(t) \in G$ for some $t \in \mathbb{N}$.

The control problem $(\mathcal{S}, I, G)$ has a **solution** if there exists a controller K such that **all runs** starting in I reach G in a **finite number of steps** in the closed loop system $\mathcal{S}^{(K)}$.

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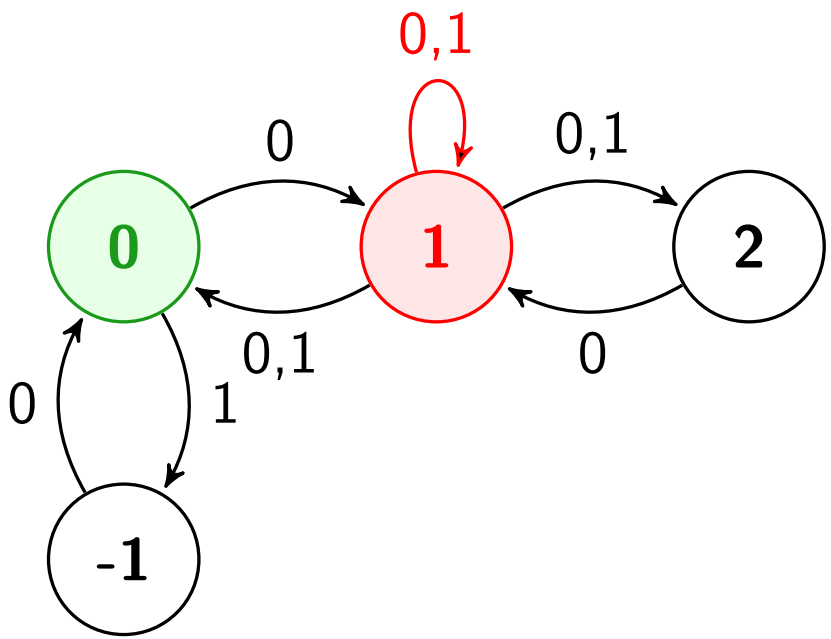
LTS Reachability Control Problem

Example (I)

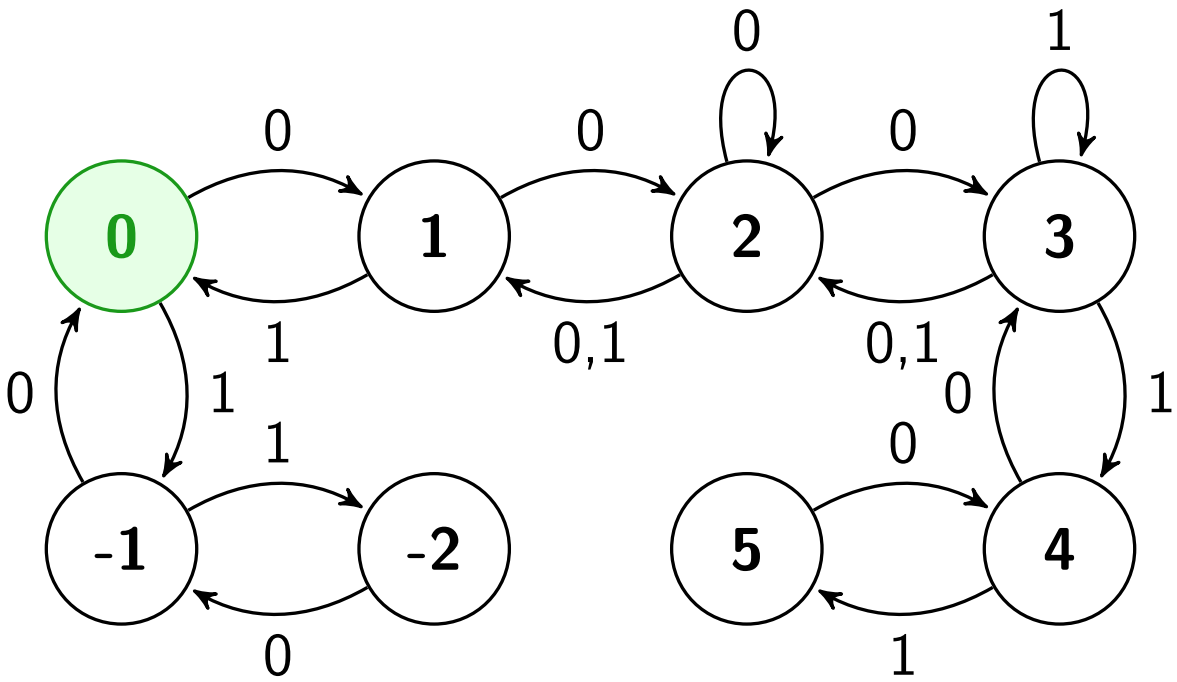
There is no quantized solution because of self–loops in state 1. The worst case distance of 0 from 1 is **infinite**.



LTS Reachability Control Problem

Example (II)

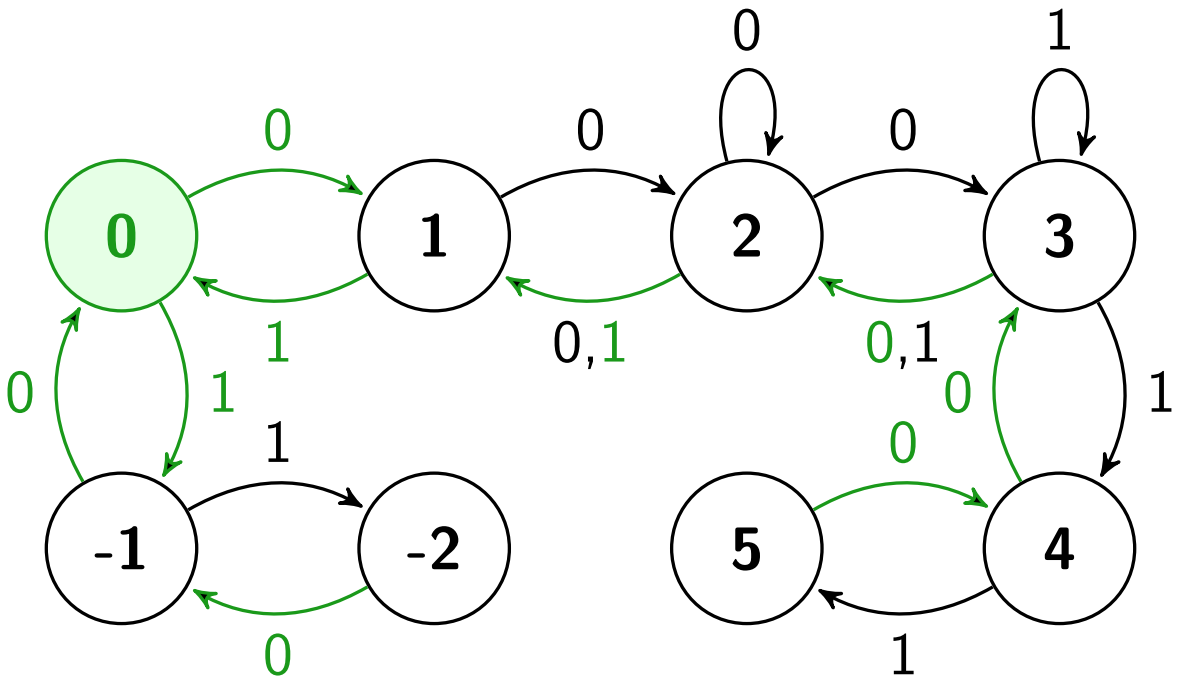
Quantized solutions exist.



Quantized Control Problem

Example (II)

The controller that enables green actions is a solution.



DTLHS Reachability and Control Problem

A **DTLHS control problem** (**reachability problem**) $(\mathcal{H}, I, G)$ is defined as the LTS control problem (reachability) $(LTS(\mathcal{H}), I, G)$.

Example. Let T be $1/10$ (sampling time). Let $\mathcal{H}$ be $(\{x\}, \{u\}, \varnothing, N)$ where: x is a continuous variable, u is a boolean variable, and

$$N(x, u, x') \equiv [\overline{u} \rightarrow x' = x + (5/4 - x)\,T] \wedge [u \rightarrow x' = x + (x - 7/4)\,T].$$

Let us consider the control problem $\mathcal{P} = (\mathcal{H}, I, G)$, where:

$$I(x) \equiv -1 \leq x \leq 5/2 \qquad \text{and} \qquad G(x) \equiv 0 \leq x \leq 1/2.$$

A **solution** K to $\mathcal{P}$ is:

$$K(x, u) = (-1 \leq x < 0 \wedge \overline{u}) \vee (0 \leq x < 3/2 \wedge u) \vee (3/2 \leq x \leq 5/2 \wedge \overline{u}).$$

Observe that $N(5/4, 0, 5/4)$ and $N(7/4, 1, 7/4)$ hold, hence no solution can enable action 0 in $5/4$ and action 1 in $7/4$.

Quantized Control Problem for DTLHSs

A **quantization function** is a non-decreasing function $\gamma_x : \mathcal{D}_x \subseteq \mathbb{R} \mapsto [a, b] \subseteq \mathbb{Z}$

The quantization of a sequence of variables is a sequence of quantization functions: $\Gamma = \{\gamma_{x_1}, \ldots, \gamma_{x_n}\} \quad \Gamma(s) = \langle \gamma_{x_1}(s_1), \ldots, \gamma_{x_n}(s_n) \rangle$

K is a **quantized controller** if there exists $\hat{K} : \Gamma(\mathcal{D}_X) \times \Gamma(\mathcal{D}_U) \to \mathbb{B}$, such that $K(s, a) = \hat{K}(\Gamma(s), \Gamma(a))$.

This enables a **software implementation** of the controller.

Quantized Control Problem

Control Abstraction

We build a finite LTS, the **control abstraction** of a DTLHS $\mathcal{H}$

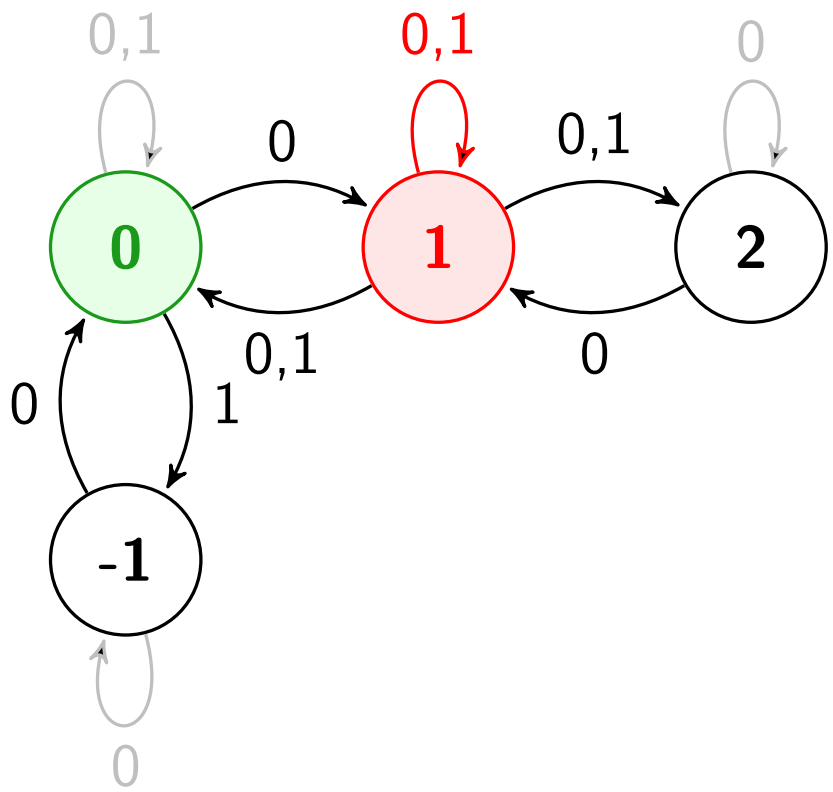
- The set of states is $\Gamma(\mathcal{D}_X)$
- The set of action is $\Gamma(\mathcal{D}_U)$
- $T(s, a, s')$ iff there exists $x \in \Gamma^{-1}(s)$, $x' \in \Gamma^{-1}(s')$, $u \in \Gamma^{-1}(a)$, $y \in \mathcal{D}_y$ such that $N(x, u, y, x')$

A self–loop $T(s, a, s)$ is **non-eliminable** if there exists an infinite run $\pi = x_0 u_0 x_1 u_1 x_2 \ldots$ in $\mathcal{H}$ such that $\forall t \in \mathbb{N}\ x_t \in \Gamma^{-1}(\hat{s})$ and $a_t \in \Gamma^{-1}(\hat{a})$.

Quantized Control Problem

Example (I)

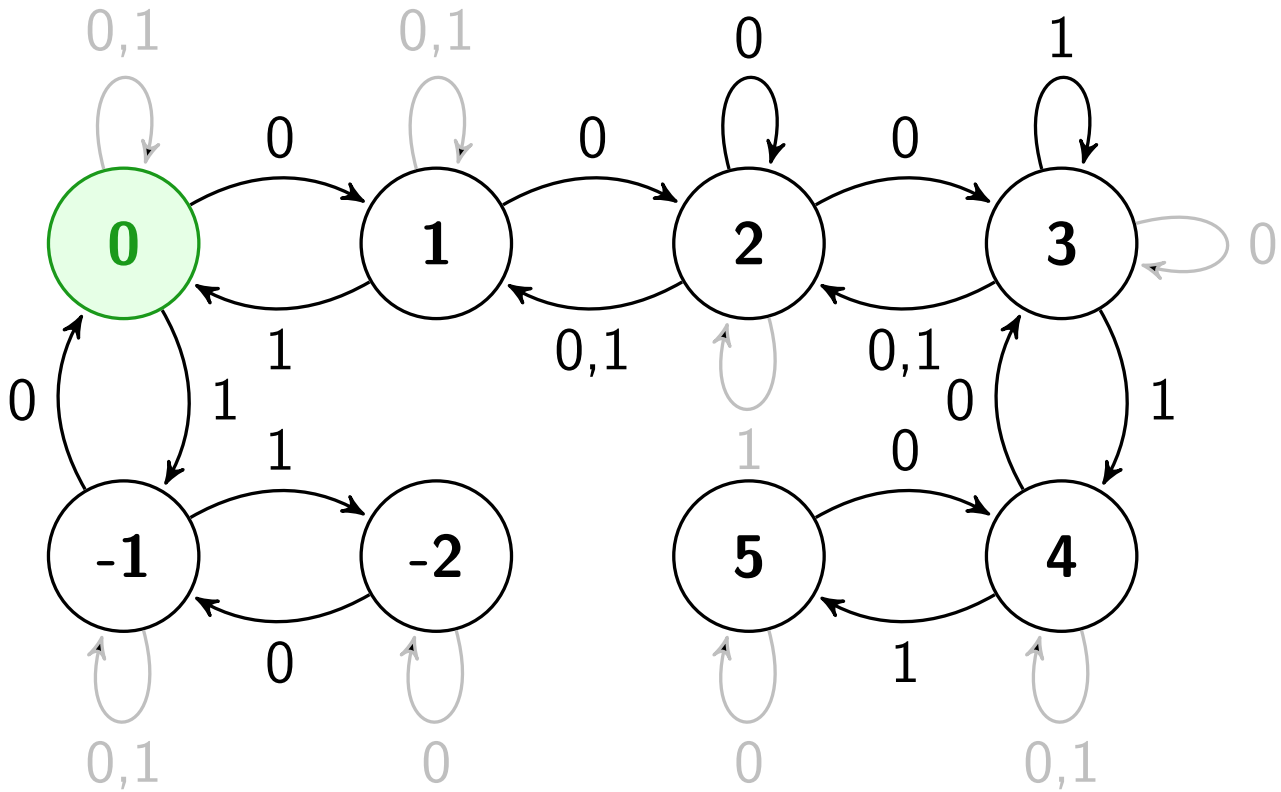
Let $\gamma_x(x) = \lfloor x \rfloor$. There is no quantized solution because of self–loops in state 1. The worst case distance of 0 from 1 is **infinite**.



Quantized Control Problem

Example (II)

Let $\gamma_x(x) = \lfloor 2x \rfloor$. Quantized solutions exist. (Light gray edges are eliminable self–loops)



Quantized Control Problem

Example (II)

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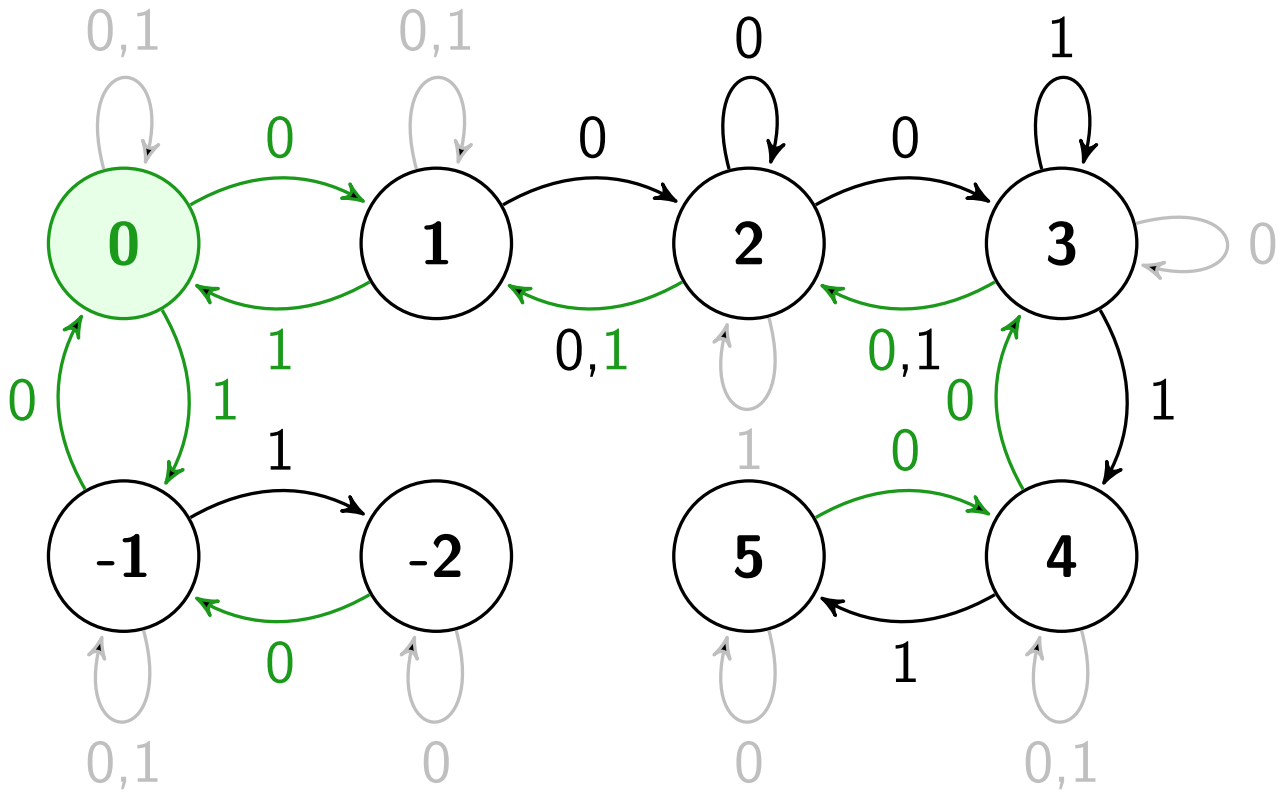

Outline

A First Attempt

inspired by [Henzinger, Kopke, Puri, Varaiya - 1996]

Reduce a known undecidable problem about Hybrid Systems to (quantized) DTLHS control problem.

Rectangular Hybrid Automata reachability can be encoded into DTLHS reachability and control problems (see **[Mari, Melatti, Salvo, Tronci, ICTAC 2012]**). This **proves undecidability of DTLHS reachability and control problem**.

This is an interesting result (enconding of a dense time model into a discrete time model). However, the **undecidability** of the DTLHS **quantized control problem** does not follow immediately.

Given a quantization schema, **the number of quantized controllers is finite**. This may lead to think that the problem might be decidable.

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Basic Idea

Reduce halting problem for Two Counter Turing Machines to DTLHS (quantized) control problem.

For any *two counter Turing Machine M* we find a DTLHS $\mathcal{H}_M$ and control problem $\mathcal{P} = (\mathcal{H}_M, I, G)$ such that M **halts if and only if** $\mathcal{P}$ **has a solution**.

In our encoding, $\mathcal{H}_M$ **has no controllable actions**, and the undecidability of the quantized control problem for DTLHSs trivially follows.

Two Counter Turing Machines

[Minsky 1961]

Two-counter Turing Machines are a *minimal* **Turing-complete** model of computation. They consist of:

- two counters that store unbounded natural numbers
- finite control program $\langle 1 : stmt_1, \ldots, n : stmt_n \rangle$ where:

$$stmt ::= \text{inc } i \; k \mid \text{dec } i \; k \mid \text{beq } i \; k \mid \text{halt} \qquad (i \in \{0, 1\})$$

- Example of computation
  - $j : \text{beq } i \; k \rightarrow \begin{cases} \text{goto statemet labeled } k \text{ if counter } i \text{ stores } 0 \\ \text{goto statemet labeled } k+1 \text{ otherwise} \end{cases}$
  - $j : \text{inc } i \; k \rightarrow$ add 1 to counter i and goto statemet labeled k

Encoding a Two Counter TM into a DTLHS (I)

Given a two-counter machine M let $\mathcal{H}_M$ be the DTLHS (X, U, Y, N), where:

- $X^r = \{x_0, x_1\}$, $\mathcal{D}_{x_i} = [0, 1]$
  (**idea:** if counter i stores the natural n, x_i assumes value $1/2^n$)
- $X^d = \{l, g\}$
  (**idea**: l stores labels of statements, g is 1 iff the computation halts)
- $U = Y = \varnothing$.

Encoding a Two Counter TM into a DTLHS (II)

Transition Relation

A program $\langle 1 : stmt_1, \ldots, n : stmt_n \rangle$ is encoded by the predicate $N = \bigwedge_{j=1}^{n} [\![j : stmt_j]\!]$, where:

$$
\begin{aligned}
[\![j : \text{dec } i \; k]\!] &\equiv (l \neq j) \vee (((x_i = 1) \vee (x_i' = 2x_i)) \wedge \\
&\quad \wedge ((x_i \neq 1) \vee (x_i' = 1)) \wedge (l' = k) \wedge U(x_{1-i}, g)) \\
[\![j : \text{inc } i \; k]\!] &\equiv (l \neq j) \vee ((x_i' = {}^{x_i}/_2) \wedge (l' = k) \wedge U(x_{1-i}, g)) \\
[\![j : \text{beq } i \; k]\!] &\equiv (l \neq j) \vee (((x_i \neq 1) \vee (l' = k)) \wedge \\
&\quad \wedge ((x_i = 1) \vee (l' = l + 1)) \wedge U(x_{1-i}, g)) \\
[\![j : \text{halt}]\!] &\equiv (l \neq j) \vee ((l' = j) \wedge (g' = 1) \wedge U(x_0, x_1))
\end{aligned}
$$

Main Results

Lemma. For any two-counter machine M, there exists a *bounded*, *conjunctive*, and *deterministic* DTLHS $\mathcal{H}_M$, and two predicates I and G such that M halts if and only if G is reachable from I in $\mathcal{H}_M$.

Theorem. The reachability problem for bounded and conjunctive DTLHSs is undecidable.

Theorem. Existence of solutions to a control problem for a bounded and conjunctive DTLHS is undecidable.

Theorem. Existence of quantized solutions to a DTLHS control problem is undecidable.

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Outline

Conclusion and Future Work

Conclusions

- We have shown that, for DTLHSs, existence of a quantized sampling controller meeting given (safety and liveness) system level specifications is undecidable.
- we have shown that *Rectangular Automata* (RA), and thus *Timed Automata* (TA), can be modelled as DTLHSs.

Future Work

- Investigating interesting classes of **discrete time** hybrid systems for which quantizing sampling control is decidable: however this can lead to consider **not enough expressive models**.
- Find **easy to compute sufficient conditions** (resp. **necessary conditions**) for the existence (resp. non existence) of a quantized controller.

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Good News!

for smart "young" people

- **Post-Doc positions available in the Model Checking Group in Rome** within 2 FP7 european projects.
- **Topics**: application of control synthesis to Energy distribution (SmartHG) and Human Fertility (PAEON).
- **Not exactly TCS**, but people involved in Formal Methods are welcome to apply (Remember: practice can inspire excellent theorectical work!).

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Any questions?

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