Time Modalities over Many-valued Logics

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Fuzzy logic

- Fuzzy Logic is a logical system which is an extension of multivalued logic and is intended to serve, as a logic of approximate reasoning

Fuzzy Logic vs. Probability

- Fuzzy logic
  - It deals with not measurable events
  - The definition of the considered events is vague
  - Ex.: Tomorrow will be cold
- Probability
  - It deals with observable events whose occurrence is uncertain
  - Ex.: Tomorrow the temperature will be $10°$C at 12:00

From crisp to fuzzy connectives

The semantics of existing fuzzy temporal operators is based on the idea of replacing classical connectives or propositions with their fuzzy counterparts.

- Fuzzy LTL (FLTL) [Lamine, Kabanza]:
  LTL in which Boolean operators are interpreted as in Zadeh interpretation

Do not allow to represent additional temporal properties, such as almost always, soon.

From fuzzy connectives to fuzzy modalities

Introduction of proper fuzzy temporal operators to represent short/long time distance in which a specific property must be satisfied

- Lukasiewicz TL (FLTL) [Thiele, Kalenka]:
  LTL with short/medium/long term operators

No specific fuzzy semantics for temporal modalities: depend on the interpretation given to a (sub-)argument, which is an untimed fuzzy formula.

FTL: Fuzzy Time modalities in LTL

- We want to add temporal modalities such as “often”, “soon”, etc. This kind of modalities may be useful when we need to specify situations when a formula is slightly satisfied, since an event happens a little bit later than expected, when a property is always satisfied except for a small set of time instants, or a property is maintained for a time interval which is slightly smaller than the one.
- The underlying logic is a t-norm based logic.

t-norm/conorm, implication $\&$ negation

	boundary value	commutativity	associativity	monotonicity
negation	$\ominus 0 = 1$ $\ominus 1 = 0$	-	-	$\alpha \leq \beta \Rightarrow \ominus\alpha \geq \ominus\beta$
t-norm	$\alpha \otimes 0 = 0$ $\alpha \otimes 1 = \alpha$	yes	yes	$\beta \geq \gamma \Rightarrow \alpha \otimes \beta \geq \alpha \otimes \gamma$ $\alpha \otimes \beta \leq \alpha$
t-conorm	$\alpha \oplus 0 = \alpha$ $\alpha \oplus 1 = 1$	yes	yes	$\beta \geq \gamma \Rightarrow \alpha \oplus \beta \geq \alpha \oplus \gamma$ $\alpha \oplus \beta \geq \alpha$
implication	$1 \circleddash \beta = \beta$ $0 \circleddash \beta = \alpha \circleddash 1 = 1$ $\alpha \circleddash 0 = \ominus\alpha$	no	no	$\alpha \leq \beta \Rightarrow \alpha \circleddash \gamma \geq \beta \circleddash \gamma$ $\beta \leq \gamma \Rightarrow \alpha \circleddash \beta \leq \alpha \circleddash \gamma$ $\alpha \circleddash \beta \geq \max\{\ominus\alpha, \beta\}$

Zadeh logic $\&$ t-norm based logics

	Zadeh	Gödel-Dummett	Łukasiewicz	Product
$\ominus\alpha$	$1-\alpha$	$\begin{cases} 1, & \alpha = 0 \\ 0, & \alpha > 0 \end{cases}$	$1-\alpha$	$\begin{cases} 1, & \alpha = 0 \\ 0, & \alpha > 0 \end{cases}$
$\alpha \otimes \beta$	$\min\{\alpha, \beta\}$	$\min\{\alpha, \beta\}$	$\max\{\alpha+\beta-1, 0\}$	$\alpha \cdot \beta$
$\alpha \oplus \beta$	$\max\{\alpha, \beta\}$	$\max\{\alpha, \beta\}$	$\min\{\alpha+\beta, 1\}$	$\alpha+\beta-\alpha\cdot\beta$
$\alpha \circledast \beta$	$\max\{1-\alpha, \beta\}$	$\begin{cases} 1, & \alpha \leq \beta \\ \beta, & \alpha > \beta \end{cases}$	$\min\{1-\alpha+\beta, 1\}$	$\begin{cases} 1, & \alpha \leq \beta \\ \beta/\alpha, & \alpha > \beta \end{cases}$

Syntax

$$\varphi := p \mid \neg\varphi \mid \varphi \sim \varphi \mid \mathcal{O}\varphi \mid \varphi\mathcal{T}\varphi$$

- Unary modalities
  - $\mathcal{F}$, $(\mathcal{F}_t)$ eventually
  - $\mathcal{G}$, $(\mathcal{G}_t)$, $\mathcal{AG}$, $(\mathcal{AG}_t)$ globally & almost globally (or often)
  - $\mathcal{X}$, $\mathcal{S}$*oon* next & soon
  - $\mathcal{W}_t$, $\mathcal{L}_t$ within & lasts t instants
- Binary modalities
  - $\mathcal{U}$, $(\mathcal{U}_t)$, $\mathcal{AU}$, $(\mathcal{AU}_t)$ until & almost until

Fuzzy satisfiability

- It is defined w.r.t. a linear structure (S, s_0, π, L)
- An strictly decreasing *avoiding function* $\eta : \mathbb{Z} \to [0,1]$: $\eta(i) = 1$, $\forall i \leq 0$, and $\eta(n_\eta) = 0$ for some $n_\eta \in \mathbb{N}$.
- *Fuzzy satisfiability relation* $\models \;\subseteq\; S^\omega \times F \times [0,1]$, where $(\pi \models \varphi) = \nu \in [0,1]$ means that the truth degree of φ along π is ν.

Connectives

- t-norm substitutes $\wedge$
- t-conorm substitutes $\vee$

$$
\begin{aligned}
&(\pi^i \models p) = L(s^i)(p),\\
&(\pi^i \models \neg\varphi) = \ominus(\pi^i \models \varphi),\\
&(\pi^i \models \varphi \wedge \psi) = (\pi^i \models \varphi) \otimes (\pi^i \models \psi),\\
&(\pi^i \models \varphi \vee \psi) = (\pi^i \models \varphi) \oplus (\pi^i \models \psi),\\
&(\pi^i \models \varphi \Rightarrow \psi) = (\pi^i \models \varphi) \otimes\!\!\!> (\pi^i \models \psi),
\end{aligned}
$$

Next and Soon

- $\mathcal{X}$ has the same semantics of its corresponding LTL operator :

$$(\pi^i \models \mathcal{X}\varphi) = (\pi^{i+1} \models \varphi).$$

- $\mathcal{S}\mathit{oon}$ extends $\mathcal{X}$ by tolerating at most n_η time instants of delay:

$$(\pi^i \models \mathcal{S}\mathit{oon}\,\varphi) = \bigoplus_{j=i+1}^{i+n_\eta} (\pi^j \models \varphi) \cdot \eta(j - i - 1).$$

- Remark:

$$(\pi^i \models \mathcal{X}\varphi) \leq (\pi^i \models \mathcal{S}\mathit{oon}\,\varphi).$$

Next and Soon: example

n	0	1	2	3	4
$\eta(n)$	1	0.73	0.69	0.26	0
$\pi^0 \models p$	1	0.51	0.75	0.99	1

$$\pi^0 \models \mathcal{S}oon\, p = 1 \cdot 0.51 \oplus 0.73 \cdot 0.75 \oplus 0.69 \cdot 0.99 \oplus 0.26 \cdot 1$$
$$= \begin{cases} 0.6831 & \text{(Z)} \\ 1 & \text{(Ł)} \\ \sim 0.928 & (\Pi) \end{cases}$$

Eventually

- $\mathcal{F}$ and $\mathcal{F}_t$ maintain the semantics of LTL operator $\mathbf{F}$:

$$(\pi^i \models \mathcal{F}_t\varphi) = \bigoplus_{j=i}^{i+t}(\pi^j \models \varphi),$$

$$(\pi^i \models \mathcal{F}\varphi) = \bigoplus_{j\geq i}(\pi^j \models \varphi) = \lim_{t\to+\infty}(\pi^i \models \mathcal{F}_t\varphi).$$

- Remark: $\mathcal{F}$ is well defined by monotonicity and if $t \leq t'$:

$$(\pi^i \models \varphi) \leq (\pi^i \models \mathcal{F}_t\varphi) \leq (\pi^i \models \mathcal{F}_{t'}\varphi) \leq (\pi^i \models \mathcal{F}\varphi).$$

Within

- $\mathcal{W}_t$ is inherently bounded:

$$(\pi^i \models \mathcal{W}_t\varphi) = \bigoplus_{j=i}^{i+t+n_\eta-1} (\pi^j \models \varphi) \cdot \eta(j-t-i).$$

- $\mathcal{W}_t p$ means p is supposed to hold in at least one of the next t instant or, possibly, in the next $t+n_\eta$. In the last case we apply a penalization for each instant after the t-th.
- Remark

$$\mathcal{W}_0\varphi \equiv \mathcal{S}oon\,\varphi$$
$$\mathcal{W}_t\varphi \equiv \mathcal{F}_t\varphi \vee \mathcal{X}^{t+1}\mathcal{S}oon\,\varphi$$
$$(\pi^i \models \mathcal{W}_t\varphi) \geq (\pi^i \models \mathcal{F}_t\varphi)$$
$$\lim_{t\to+\infty}(\pi^i \models \mathcal{W}_t\varphi) = (\pi^i \models \mathcal{F}\varphi)$$

Always

- $\mathcal{G}$ and $\mathcal{G}_t$ extend the semantics of $\mathbf{G}$:

$$(\pi^i \models \mathcal{G}_t\varphi) = \bigotimes_{j=i}^{i+t}(\pi^j \models \varphi),$$
$$(\pi^i \models \mathcal{G}\varphi) = \bigotimes_{j\geq i}(\pi^j \models \varphi) = \lim_{t\to+\infty}(\pi^i \models \mathcal{G}_t\varphi).$$

- Remark: $\mathcal{G}$ is well defined by monotonicity and if $t \leq t'$:

$$\begin{aligned}(\pi^i \models \mathcal{G}\varphi) &\leq (\pi^i \models \mathcal{G}_t\varphi) \leq (\pi^i \models \mathcal{G}_{t'}\varphi)\\ &\leq (\pi^i \models \mathcal{G}_1\varphi) = (\pi^i \models \varphi \wedge \mathcal{X}\varphi)\\ &\leq (\pi^i \models \mathcal{G}_0\varphi) = (\pi^i \models \varphi)\end{aligned}$$

Almost always (Often)

- $\mathcal{AG}$ and $\mathcal{AG}_t$ evaluate a property over a path π^i, by avoiding at most n_η evaluations of this property, and introducing a penalization for each avoided case.
- Let I_t be the initial segment of $\mathbb{N}$ of length $t+1$ and $\mathcal{P}^k(I_t)$ the set of subsets of I_t of cardinality k:

$$(\pi^i \models \mathcal{AG}_t\, \varphi) = \max_{j \in I_t} \max_{H \in \mathcal{P}^{t-j}(I_t)} \bigotimes_{h \in H} (\pi^{i+h} \models \varphi) \cdot \eta(j)$$
$$(\pi^i \models \mathcal{AG}\, \varphi) = \lim_{t \to +\infty} (\pi^i \models \mathcal{AG}_t\, \varphi)$$

- Remark: the sequence $(\pi^i \models \mathcal{AG}_t\, \varphi)_{t \in \mathbb{N}}$ is not monotonic. Still, the semantics of $\mathcal{AG}$ is well-defined.

Almost always (Often): properties

- It is possible to recursively define n propositional letters $p_0, \ldots, p_{n-1}$, such that

$$(\pi^i \models \mathcal{AG}\,\varphi) = \max_{j \leq n_\eta - 1} \{\mathcal{G}p_j \cdot \eta(j)\}$$

- Corollary: $\mathcal{AG}$ is well-defined
- $$(\pi^i \models \mathcal{AG}_t\,\varphi) \geq (\pi^i \models \mathcal{G}_t\varphi),$$
$$(\pi^i \models \mathcal{AG}\,\varphi) \geq (\pi^i \models \mathcal{G}\varphi).$$
- Remark: it is not possible to establish a priori which inequality holds between $(\pi^i \models \mathcal{AG}_t\,\varphi)$ and $(\pi^i \models \mathcal{AG}_{t'}\,\varphi)$

Almost globally: example

n	0	1	2	3	4	5
$\eta(n)$	1	0.73	0.69	0.26	0	0
$\pi^0 \models p$	0.51	0.68	0.22	0.99	0.82	0.45

$$
\begin{aligned}
(Z): \quad \pi^0 \models \mathcal{AG}_5\, p \quad &= \max\{0.51 \oplus 0.68 \oplus 0.22 \oplus 0.99 \oplus 0.82 \oplus 0.45, \\
&= 0.73 \cdot (0.51 \oplus 0.68 \oplus 0.99 \oplus 0.82 \oplus 0.45), \\
&= 0.69 \cdot (0.51 \oplus 0.68 \oplus 0.99 \oplus 0.82), \\
&= 0.26 \cdot (0.68 \oplus 0.99 \oplus 0.82)\} \\
&= \max\{0.22, 0.3285, 0.3519, 0.1768\} = 0.3519
\end{aligned}
$$

Lasts

- $\mathcal{L}_t$ expresses that a property lasts for t consecutive instants from now, possibly avoiding some event:

$$(\pi^i \models \mathcal{L}_t\varphi) = \max_{0\leq j\leq \min\{t, n_\eta - 1\}} \{(\pi^i \models \mathcal{G}_{t-j}\varphi)\cdot\eta(j)\}.$$

- Remark:

$$(\pi^i \models \mathcal{G}_t\varphi) \leq (\pi^i \models \mathcal{L}_t\varphi) \leq (\pi^i \models \mathcal{A}\mathcal{G}_t\,\varphi)$$
$$\lim_{t\to+\infty}(\pi^i \models \mathcal{L}_t\varphi) = (\pi^i \models \mathcal{G}\varphi)$$

Until and Almost Until

- $\mathcal{U}$ and $\mathcal{U}_t$ naturally extends the corresponding LTL operator $\mathbf{U}$:

$$\begin{array}{l}(\pi^i \models \varphi\, \mathcal{AU}_0\, \psi) = (\pi^i \models \psi),\\ (\pi^i \models \varphi\, \mathcal{AU}_t\, \psi) = \max_{i \leq j \leq i+t}\left((\pi^j \models \psi) \otimes (\pi^i \models \mathcal{AG}_{j-1}\, \varphi)\right),\\ (\pi^i \models \varphi\, \mathcal{AU}\, \psi) = \lim_{t \to +\infty}(\pi^i \models \varphi\, \mathcal{AU}_t\, \psi),\end{array}$$

- Remark:

$$(\pi^i \models \varphi \mathcal{U}\, \psi) \leq (\pi^i \models \mathcal{F}\psi)$$

$$\begin{aligned}(\pi^i \models \psi) &= (\pi^i \models \varphi\, \mathcal{AU}_0\, \psi) \leq (\pi^i \models \varphi \mathcal{U}_t\, \psi)\\ &\leq (\pi^i \models \varphi\, \mathcal{AU}_t\, \psi) \leq (\pi^i \models \varphi\, \mathcal{AU}\, \psi)\end{aligned}$$

Reduction to LTL

- Remark 1: let for all $p \in AP$ and $i \in \mathbb{N}$, $\pi^i \models p \in \{0, 1\}$, and $\eta(1) = 0$. Then FTL reduces to LTL.
- Remark 2: let $p, q \in AP$ such that for all $j \geq i$, $(\pi^j \models p), (\pi^j \models q) \in \{0, 1\}$, then

$$(\pi^i \models \mathcal{F}p) = 1 \Leftrightarrow \pi^i \models \mathbf{F}p$$
$$(\pi^i \models \mathcal{G}p) = 1 \Leftrightarrow \pi^i \models \mathbf{G}p$$
$$(\pi^i \models p\,\mathcal{U}\,q) = 1 \Leftrightarrow \pi^i \models p\mathbf{U}q$$

Adequate sets

- Adequate set: set of connectives/modalities that is sufficient to equivalently express any formula of the logic.
- Ex.: $\{\mathbf{X}, \mathbf{U}, \wedge, \neg\}$ is adequate for LTL
- For $1 \leq j < n_\eta$ define $\odot^j$:

$$(\pi^i \models \odot^j \varphi) = (\pi^i \models \varphi) \cdot \eta(j).$$

Logic	Adequate set
FTL(Z)	$\wedge, \neg, \mathcal{X}, \mathcal{U}, \mathcal{AU}, \odot^1, \ldots, \odot^{n_\eta - 1}$
FTL(G)	$\wedge, \Rightarrow, \mathcal{X}, \mathcal{U}, \mathcal{AU}, \odot^1, \ldots, \odot^{n_\eta - 1}$
FTL(Ł)	$\wedge, \Rightarrow, \mathcal{X}, \mathcal{F}, \mathcal{U}, \mathcal{AU}, \odot^1, \ldots, \odot^{n_\eta - 1}$
FTL(Π)	$\wedge, \Rightarrow, \vee, \mathcal{X}, \mathcal{F}, \mathcal{G}, \mathcal{U}, \mathcal{AU}$

FTA: Fuzzy Timed Automata

- Model a system by an enriched Timed Automata (FTA):
  an automaton with a finite set of clocks, a finite set of crisp events and a finite set of variables (control variables) representing the support for fuzzy events
- Evaluation Technique:
  - Inspired by real-time model checking and reachability analysis
  - FTA is transformed into a suitable timed transition system (FTTS) and a FTL formula is evaluated on it

From FTL to Büchi Automata

- Extend the technique to represent an LTL formula into Büchi automata [Vardi, Wolper] to express FTL formulae