Algebraic Characterization of the Class of Languages recognized by Measure Only Quantum Automata

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Summary

1. Introduction to MON-1QFAs.
2. Previous results on MON-1QFA.
3. Varieties of languages, monoids and literally idempotent languages.
4. Algebraic Characterization of **LMO**(Σ).
5. More on varieties of languages and monoids.
6. Latvian Automata and block groups.
7. Proof sketch.
8. Deciding **LMO**(Σ) membership.

Measure Only Quantum Automata

The MON-1QFA model has been introduced in [BMP10]. We consider MON-1QFAs over alphabet Σ.

Definition

A MON-1QFA over the alphabet Σ is a tuple of the form

$$A = \langle \Sigma \cup \{\#\}, \pi_0, (O_c)_{c \in \Sigma \cup \{\#\}}, F \rangle$$

1. The complex m-dimensional vector $\pi_0 \in \mathbb{C}^{1 \times m}$, with unitary norm $||\pi_0|| = 1$, is called the quantum initial state of A.

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3. The subset $F \subseteq V(O_\#)$ of the eigenvalues of $O_\#$ is called the spectrum of the quantum final accepting states of A.

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1. Consider current state $\pi \in \mathbb{C}^{(1\times m)}$ and current input letter $c \in \Sigma$ taken from input word $w \in \Sigma^*$.

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6. After last input letter of w, apply $O_\#$ and obtain eigenvalue $\lambda_\#$. If $\lambda_\# \in F$, then A accepts w, otherwise it rejects it.

MON-1QFA probabilistic behaviour

Definition (BMP10)

Define density matrix $\sigma(w)$ for every $w \in \Sigma^*$ as follows

$$\sigma(w) = \begin{cases} \pi_0^\dagger \pi_0 & \text{if } w = \epsilon \\ \sum_{j=1}^{k(c)} P_j(c)\sigma(t)P_j(c) & \text{if } w = tc, c \in \Sigma \text{ and } k(c) \\ & \text{is the cardinality of } O_c\text{'s spectrum} \end{cases}$$

Definition (BMP10)

For every $w \in \Sigma^*$ the probability of MON-1QFA A accepting $w = w_1 \cdots w_n$ is given by

$$\begin{aligned} p_A(w) &= \mathrm{Tr}\left(\sum_{r_j(\#)\in F} P_j(\#)\sigma(w)P_j(\#)\right) \\ &= \sum_{r_j(\#)\in F} \sum_{j_1,\dots,j_n} ||\pi_0 P_{j_1}(w_1)\cdots P_{j_n}(w_n)P_j(\#)||_2^2 \end{aligned}$$

MON-1QFA probabilistic acceptance with isolated cut-point

Definition

Let A be a MON-1QFA, let $w \in \Sigma^*$ be an input word. Define $p_A(w)$ as the probability that A accepts w. We say that A accepts L with cut-point λ isolated by $\delta > 0$ iff

1. $w \in L$ imply $p_A(w) \geq \lambda + \delta$
2. $w \notin L$ imply $p_A(w) \leq \lambda - \delta$.

Definition

We denote by $\mathbf{LMO}(\Sigma)$ the class of languages recognized by some MON-1QFA with isolated cut-point.

Variation of formal languages

Following [BMP10],

Definition

Let $L \in \Sigma^*$ be a regular language. Let $A = \langle \Sigma, Q, \delta, q_0, F \rangle$ be the minimum deterministic automaton recognizing L. For any string $w = w_1 \cdots w_n \in \Sigma^*$ define the variation of L w.r.t. w as

$$\mathrm{Var}_L(w) := |\{k | 0 < k < n, \delta(q_0, w_1 \cdots w_k) \neq \delta(q_0, w_1, \cdots, w_{k+1})\}|$$

Definition

Let $L \in \Sigma^*$ be a regular language. We say that L has finite variation iff $\sup_{w \in \Sigma^*} \mathrm{Var}_L(w) < \infty$.

Some previous results on MO-1QFA

In [BMP10] the authors study MON-1QFAs over compatibility alphabets (Σ, E) showing the following

Theorem

LMO(Σ, E) *is a boolean algebra of recognizable languages with finite variation.*

Considering classical alphabet Σ in place of compatibility alphabets (Σ, E), the same result holds for $LMO(\Sigma)$. This is a boolean algebra of regular languages with finite variation.

Varieties of languages, monoids and literally idempotent languages

1. $L \in \Sigma^*$ is literally idempotent iff for all $x, y \in \Sigma^*$ and $a \in \Sigma$, $xa^2y \in L \iff xay \in L$, we denote this class of languages by lId.

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2. $L \in \Sigma^*$ is literally idempotent piecewise testable iff it lies in the boolean closure of the following class of languages $\Sigma^* a_1 \Sigma^* a_2 \Sigma^* \cdots \Sigma^* a_k \Sigma^*$ for $k \geq 1, a_1, \ldots, a_k \in \Sigma$ and $a_1 \neq \cdots \neq a_k$, we denote this class of languages by lild**PT**. [KP08]

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3. denote by J, L, R the Green's relations determined by double, left and right sided ideals of monoids and denote by **J** the pseudovariety of J-trivial monoids.

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3. denote by J, L, R the Green's relations determined by double, left and right sided ideals of monoids and denote by **J** the pseudovariety of J-trivial monoids.
4. denote by $\bar{\mathbf{J}}$ the class of J-trivial monoids such that for every $M \in \bar{\mathbf{J}}$ every syntactic morphism $\phi : \Sigma^* \to M$ satisfy the idempotency condition literally, i.e. $\phi(\sigma)\phi(\sigma) = \phi(\sigma)$, for every $\sigma \in \Sigma$. [KP08]

Algebraic Characterization of $LMO(\Sigma)$

In this work we defend the following,

Theorem

Let $L \subseteq \Sigma^*$ be a formal language over the alphabet Σ. Then the following three propositions are equivalent:

1. *There exists a* MON-1QFA *over Σ recognizing L with isolated cut-point.*
2. *The syntactic monoid $M(L)$ is J-trivial and literally idempotent.*
3. *L is a literally idempotent piecewise testable regular language over the alphabet Σ.*

Stated as an equation between varieties of formal languages,

$$\mathbf{LMO}(\Sigma) = \mathcal{V}_\Sigma(\bar{\mathbf{J}}) = \text{lild}\mathbf{PT}(\Sigma)$$

More on varieties of languages, monoids and literally idempotent languages

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4. Still, it is closed under inverse literal morphisms, i.e. morphisms $\phi : \Gamma^* \to \Sigma^*$ s.t. $\phi(\Gamma) \subseteq \Sigma$.
5. Under this perspective, $V(\mathbf{J}) \cap \mathsf{lild}$ has been studied and characterized in [KP08] as the class of languages corresponding to $\overline{\mathbf{J}}$ monoids.

A theorem of Klima and Polak

Recall that lild**PT** is defined as the boolean closure of langauges $\Sigma^* a_1 \Sigma^* a_2 \cdots \Sigma^* a_n \Sigma^*$ for $a_i \neq a_{i+1}$. Then,

Theorem (KP08)

Let $L \subseteq \Sigma^*$ be a formal language. The following propositions are equivalent:

1. $L \in lild\mathbf{PT}(\Sigma)$
2. $L \in V_\Sigma(\mathbf{J}) \cap lild$
3. $L \in V_\Sigma(\bar{\mathbf{J}})$.

Algebraic characterization of Latvian Automata

Definition

A monoid is a block group iff every R-class and every L-class has at most one idempotent element.

Definition (Amb06)

A Latvian automaton (LQFA) is a tuple of the following form

$$\langle Q, \Sigma, \{U_\sigma\}, \{O_\sigma\}, q_0, F\rangle$$

1. every U_σ is unitary, $U_\sigma^{-1} = U_\sigma^\dagger$.
2. every O_σ is Hermitian.

Algebraic characterization of Latvian Automata

Theorem (Amb06)

Let L be a regular language. There exists a LQFA recognizing L with isolated cut-point iff the syntactic monoid of L is a block group.

Algebraic Characterization of **LMO**, Proof sketch

The proof is composed by two directions, we denote $(\Rightarrow)$ and $(\Leftarrow)$.

1. $(\Rightarrow)$ by combining the theorem of Bertoni, Mereghetti, Palano [BMP10], the algebraic characterization of Latvian Automata of Ambainis [Aal06], and a theorem of Klima and Polak [KP08], we see that every language recognized by a MON-1QFA with isolated cut-point is a literal idempotent piecewise testable language.
2. $(\Leftarrow)$ we explicitly exhibit a family of orthogonal projection operators, that allow us to recognize with isolated cut-point every literally idempotent piecewise testable language by means of MON-1QFAs.

Finite variation is equivalent to R-triviality Lemma

Theorem

Let L be a regular formal language. The following propositions are equivalent.

1. *L has finite variation.*
2. *Every strongly connected compotent of the minimum automaton of L has exactly one vertex.*
3. *Let S be the set of states of the minimum automaton recognizing L. There exists a total ordering of S such that, for every $q \in Q$ and every $a \in \Sigma$, $qa \geq q$.*
4. *The syntactic monoid of L is an R-trivial monoid.*

Algebraic Characterization of **LMO**, Proof sketch of ($\Rightarrow$)

1. the class of MON-1QFA over Σ is a sub-class of Latvian Automata, fully characterized algebraically by the class **BG** of block groups.

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4. show that every R-trivial block group monoid is a J-trivial monoid.
5. deduce $\mathbf{LMO}(\Sigma) \subseteq V_\Sigma(\mathbf{J})$, observe further that every language in $\mathbf{LMO}(\Sigma)$ is literally idempotent and apply the theorem of Klima and Polak. This imply $\mathbf{LMO}(\Sigma) \subseteq V_\Sigma(\bar{\mathbf{J}}) = \text{lild}\mathbf{PT}(\Sigma)$.

Proof sketch of ($\Leftarrow$)

1. Consider the language

$$L[a_1, \dots, a_k] = \Sigma^* a_1 \Sigma^* \cdots \Sigma^* a_k \Sigma^*$$

where $a_1, \dots, a_k \in \Sigma$, $a_i \neq a_{i+1}$ for $1 \leq i < k$, and let $S = \{a_1, \dots, a_k\}$.

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3. Let $j_1^{(\alpha)} < j_2^{(\alpha)} < \cdots < j_{\#\alpha}^{(\alpha)}$ be all the indexes such that

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4. We will define, for every $\alpha \in S$, two orthogonal projectors of dimension $(k+1) \times (k+1)$: the up operator $P^{(k)}_{\nearrow}(\alpha)$ and the down operator $P^{(k)}_{\searrow}(\alpha)$...

Proof sketch of $(\Leftarrow)$

Definition

We define elementary operators $P_{\nearrow}$ and $P_{\searrow}$ of dimension two as follows

$$P_{\nearrow} := \begin{bmatrix} +\frac{1}{2} & +\frac{1}{2} \\ +\frac{1}{2} & +\frac{1}{2} \end{bmatrix}$$

$$P_{\searrow} := \begin{bmatrix} +\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & +\frac{1}{2} \end{bmatrix}$$

Definition

Using $P_{\nearrow}$ we will define $P^{(k)}_{\nearrow}(\alpha)$ and then using and $P_{\searrow}$ we will define $P^{(k)}_{\searrow}(\alpha)$... as follows ...

$P^{(k)}_{\nearrow}(\alpha) =$

	$1,\dots$	$j_1^{(\alpha)}, j_1^{(\alpha)}+1$	$\dots$	$j_2^{(\alpha)}, j_2^{(\alpha)}+1$	$\dots$	$\dots$	$\dots$	$j_{\#\alpha}^{(\alpha)}, j_{\#\alpha}^{(\alpha)}+1$	$\dots, k+1$
$1,\dots$	I								
$j_1^{(\alpha)}$ $j_1^{(\alpha)}+1$		$P_{\nearrow}$							
$\dots$			I						
$j_2^{(\alpha)}$ $j_2^{(\alpha)}+1$				$P_{\nearrow}$					
$\dots$					I				
$\vdots$						$\ddots$			
$\dots$							I		
$j_{\#\alpha}^{(\alpha)}$ $j_{\#\alpha}^{(\alpha)}+1$								$P_{\nearrow}$	
$\dots, k+1$									I

$P^{(k)}_{\searrow}(\alpha) =$

	$1,\dots$	$j_1^{(\alpha)}, j_1^{(\alpha)}+1$	$\dots$	$j_2^{(\alpha)}, j_2^{(\alpha)}+1$	$\dots$	$\dots$	$\dots$	$j_{\#\alpha}^{(\alpha)}, j_{\#\alpha}^{(\alpha)}+1$	$\dots, k+1$
$1,\dots$	$\mathbf{0}$								
$j_1^{(\alpha)}$ $j_1^{(\alpha)}+1$		$P_{\searrow}$							
$\dots$			$\mathbf{0}$						
$j_2^{(\alpha)}$ $j_2^{(\alpha)}+1$				$P_{\searrow}$					
$\dots$					$\mathbf{0}$				
$\vdots$						$\ddots$			
$\dots$							$\mathbf{0}$		
$j_{\#\alpha}^{(\alpha)}$ $j_{\#\alpha}^{(\alpha)}+1$								$P_{\searrow}$	
$\dots, k+1$									$\mathbf{0}$

Proof sketch ($\Leftarrow$)

Definition

By calling e_j the boolean row vector such that $(e_j)_i = 1 \Leftrightarrow i = j$, we define $A[a_1, \ldots, a_k] = \langle \Sigma \cup \{\#\}, \pi_0^{(k)}, \{O_\sigma^{(k)}\}_{\sigma \in \Sigma \cup \{\#\}}, F^{(k)} \rangle$ as the MON-1QFA where

- $\pi_0^{(k)} = e_1 \in \mathbb{C}^{1 \times (k+1)}$,
- for $\alpha \in S$, the associated projectors of $O_\alpha^{(k)}$ are $P_{\nearrow}^{(k)}(\alpha)$ and $P_{\searrow}^{(k)}(\alpha)$,
- with each $O_\sigma^{(k)}$ such that $\sigma \in \Sigma \setminus S$, we associate the projector $I_{(k+1)\times(k+1)}$,
- the projector of the accepting result of $O_\#^{(k)}$ is $(e_{k+1})^T e_{k+1}$, i.e. the $(k+1) \times (k+1)$ boolean matrix having a 1 only in the bottom right entry.

Proof sketch ($\Leftarrow$)

Theorem

The automaton $A[a_1, \ldots, a_k]$ recognizes $L[a_1, \ldots, a_k]$ with cutpoint $\lambda = \frac{1}{2^{2k+1}}$ isolated by $\delta = \frac{1}{2^{2(k+1)}}$.

Proof.

$t \in L[a_1, \ldots, a_k]$ imply

$$\begin{aligned}
p_A(t) &= \sum_{r_j(\#)\in F} \sum_{j_1,\ldots,j_n} ||\pi_0 P_{j_1}(t_1) \cdots P_{j_n}(t_n) P_j(\#)||_2^2 \\
&= \sum_{j_1,\ldots,j_n \in \{\nearrow,\searrow\}} ||\pi_0 P_{j_1}(t_1) \cdots P_{j_n}(t_n) P_{\text{acc}}(\#)||_2^2 \\
&\geq ||\pi_0 P_{\nearrow}(t_1) \cdots P_{\nearrow}(t_n) P_{\text{acc}}(\#)||_2^2 \\
&= \left(\left(\pi_0 P_{\nearrow}(t_1) \cdots P_{\nearrow}(t_n) \right)_{k+1} \right)^2 \\
&\geq 2^{-2k}
\end{aligned}$$

$t \notin L[a_1, \ldots, a_k]$ imply $p_A(t) = 0$. □

Deciding **LMO** membership in time $O((|Q| + |\Sigma|)^2)$

Theorem

Given a regular language $L \in \Sigma^*$, the problem of determining whether $L \in \mathbf{LMO}(\Sigma)$ is decidable in time $O((|Q| + |\Sigma|)^2)$, where $|Q|$ is the size of the minimal deterministic automaton for L.

Proof.

This algorithm first constructs the minimal deterministic automaton A_L for L in time $O(|Q| \log(|Q|))$ as shown in [H71]. Then, in time $O(|Q| + |\Sigma|)$, it checks whether L is literally idempotent by visiting all the vertices in the graph of A_L. Finally, it verifies whether L is piecewise testable in time $O((|Q| + |\Sigma|)^2)$ with the technique shown in [T01]. The fact that $\mathbf{LMO}(\Sigma) = \mathrm{liId}\mathbf{PT}(\Sigma)$ completes the proof sketch. □

Thank you!

Thank you for your attention!

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