h-quasi planar Drawings of Bounded Treewidth Graphs in Linear Area *

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Abstract. We study the problem of computing h-quasi planar drawings in linear area; in an h-quasi planar drawing the number of mutually crossing edges is bounded by a constant h. We prove that every n-vertex partial k-tree admits a straight-line h-quasi planar drawing in $O(n)$ area, where h depends on k but not on n. For specific sub-families of partial k-trees, we present ad-hoc algorithms that compute h-quasi planar drawings in linear area, such that h is significantly reduced with respect to the general result.

1 Introduction

Area requirement of graph layouts is a widely studied topic in Graph Drawing and Geometric Graph Theory. Many asymptotic bounds have been proven for a variety of graph families and drawing styles. One of the most fundamental results in this scenario establishes that every planar graph admits a planar straight-line grid drawing in $O(n^2)$ area and that this bound is worst-case optimal [5]. This has motivated a lot of work devoted to discover sub-families of planar graphs that admit planar straight-line drawings in $o(n^2)$ area. Unfortunately, sub-quadratic upper bounds are known only for trees [4] and outerplanar graphs [6], while super-linear lower bounds are known for series-parallel graphs [13].

Although planarity is one of the most desirable properties when drawing a graph, many real-world graphs are in fact non-planar. Furthermore, planarity often imposes severe limitations on the optimization of the drawing area, which may sometimes be overcome by allowing either "few" edge crossings or specific types of edge crossings that do not affect too much the drawing readability. So far, only a few papers have focused on computing non-planar layouts in sub-quadratic area. Wood proved that every k-colorable graph admits a non-planar straight-line grid drawing in linear area [15], which implies that planar graphs admit such a drawing. However, the technique by Wood does not provide any guarantee on the type and number of edge crossings. More recently, Angelini *et al.* provided techniques for constructing poly-line *large angle crossing drawings* (*LAC drawings*) of planar graphs in sub-quadratic area [1]. We recall that the study of drawings with large angle crossings started in [9].

In this paper we study the problem of computing linear area straight-line drawings of graphs with controlled *crossing complexity*, i.e., drawings where some types of edge

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crossings are forbidden. We study *h-quasi planar drawings*, i.e., drawings with at most $h-1$ mutually crossing edges; this measure of crossing complexity can be regarded as a sort of planarity relaxation. The combinatorial properties of h-quasi planar drawings have been widely investigated [12, 14]. The contributions of the paper are as follows: (i) We prove that every n-vertex partial k-tree (i.e., any graph with bounded treewidth) admits a straight-line h-quasi planar drawing in $O(n)$ area, where h depends on k but not on n (Section 3). (ii) For specific sub-families of partial k-trees (outerplanar graphs, flat series-parallel graphs, and proper simply-nested graphs), we can compute h-quasi planar drawings in $O(n)$ area with values of h significantly smaller than those obtained with the general technique (Section 4).

For reasons of space, all the proofs and technicalities are omitted.

2 Preliminaries

A *drawing* Γ of a graph G maps each vertex v of G to a point p_v on the plane, and each edge $e = (u, v)$ to a Jordan arc connecting p_u and p_v not passing through any other vertex; furthermore, any two edges have at most one point in common. If all edges are mapped to straight-line segments, Γ is called a *straight-line drawing* of G. If all vertices are mapped to points with integer coordinates, Γ is called a *grid drawing* of G. The *bounding box* of a straight-line grid drawing Γ is the minimum axis-aligned box containing the drawing. If the bounding box has side lengths $X-1$ and $Y-1$, then we say that Γ is a drawing with *area* $X \times Y$. A drawing Γ is *h-quasi planar* if it has less than h mutually crossing edges. A 3-quasi planar drawing is also called a *quasi planar drawing*.

For definitions about track layouts see Dujmović, Pór and Wood [11].

A *k-tree*, $k \in \mathbb{N}$, is defined as follows. The clique of size k is a k-tree; the graph obtained from a k-tree by adding a new vertex adjacent to each vertex of a clique of size k is also a k-tree. A *partial k-tree* is a subgraph of a k-tree. A graph has bounded *treewidth* if and only if it is a partial k-tree [3].

3 Compact h-quasi Planar Drawings of Partial k-trees

Lemma 1. *Let G be a graph with n vertices. If G admits a (c,t)-track layout, then G admits an h-quasi planar grid drawing in $O(t^3 n)$ area, where $h = c(t-1)+1$.*

Lemma 1 implies that every graph with constant track number admits an h-quasi planar grid drawing in linear area with h being a constant. Since it is known that partial k-trees have track number that is constant in n (although depending on k) [10], this implies that every partial k-tree admits an h-quasi planar grid drawing in linear area where the value of h does not depend on n. The current best upper bound on the track number of k-trees is given in [8]. Thus, every k-tree has an h_k-quasi planar drawing in $O(n)$ area with $h_k \in O(1)$. In order to improve the value of h_k, we exploit an algorithm that computes $(2,t)$-track layouts of k-trees, whose description is omitted.

Theorem 1. *Every partial k-tree with n vertices admits an h_k-quasi planar grid drawing in $O(t_k^3 n)$ area, where $h_k = 2t_k - 1$ and t_k is given by the following set of equations:*

$$
\begin{aligned}
& t_k = (c_{k-1,k} + 1)t_{k-1} \\
& c_{k,i} = (c_{k-1,k} + 1)(c_{k-1,i} + \frac{c_{k-1,k}}{4} \sum_{j=1}^{i-1} c_{k-1,j} \cdot c_{k-1,i-j}) \quad (i = 1, \ldots, k+1) \\
& c_{k,k+2} = 0
\end{aligned}
\tag{1}
$$

with $t_1 = 2$ and $c_{1,1} = 4$ and $c_{1,2} = 2$.

We can prove that the values of h_k given in Theorem 1 are smaller than those obtained by using the track number upper bound in [8]. In particular, every partial 2-tree (i.e., every series-parallel graph) admits an 11-quasi planar drawing in $O(n)$ area.

4 Improved Bounds for Specific Families of Planar Partial k-trees

It is known that outerplanar graphs are partial 2-trees [3]. We can prove that the value of h can be reduced from 11 to 3 for outerplanar graphs.

Theorem 2. *Every outerplanar graph with n vertices admits a quasi planar grid drawing in $O(n)$ area.*

A series-parallel graph, or SP-graph, is *flat* if it does not contain two nested parallel components. For an exact definition of flat SP-graphs and decomposition tree see [7]. We lower the value of h for flat SP-graphs from 11 to 5.

Theorem 3. *Every flat SP-graph with n vertices admits a 5-quasi planar grid drawing in $O(n)$ area.*

A *proper simply-nested graph* is a k-outerplanar graph such that the vertices of levels from 1 to k are chordless cycles [2]. It is known that k-outerplanar graphs have treewidth at most $3k - 1$ [3]. By using the technique of Section 3 we would obtain an h-quasi planar drawing in linear area with h that would be a function of the number of levels k. We show that h can be reduced to 3. We remark that proper simply-nested graphs may require quadratic area if we want a planar drawing.

Theorem 4. *Every proper simply-nested graph with n vertices admits a quasi planar grid drawing in $O(n)$ area.*

5 Concluding Remarks and Open Problems

In this paper we studied the problem of computing compact h-quasi planar drawings of partial k-trees. Indeed, our algorithms can be regarded as drawing techniques that produce drawings with optimal area and with bounded crossing complexity. This point of view is particularly interesting in the case of planar graphs. As recalled in the introduction, planar graphs can be drawn with either optimal crossing complexity (i.e., in a planar way), in which case they may require $\Omega(n^2)$ area [5], or with optimal $\Theta(n)$ area but without any guarantee on the crossing complexity [15]. These two extremal results naturally raise the following question: is it possible to compute an h-quasi planar drawing of a planar graph in $o(n^2)$ area and $h \in o(n)$? In Section 4 we showed that $O(n)$ area and $h \in O(1)$ can be simultaneously achieved for some families of planar graphs. In fact Lemma 1 combined with some known results can be used to give a positive answer to the above question even for general planar graphs.

Theorem 5. *Every planar graph with n vertices admits a $O(\log^{16} n)$-quasi planar grid drawing in $O(n \log^{48} n)$ area.*

The results in this paper give rise to several interesting open problems. Among them: (1) Reducing the value of h_k given by Equation 1 for other sub-families of partial k-trees. (2) Studying whether planar graphs admits h-quasi planar drawings in $O(n)$ area with $h \in o(n)$, possibly $h \in O(1)$. (3) Studying h-quasi planar drawings in linear area and aspect ratio $o(n)$.

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